for three typical runs: one wafer polished and inspected on both sides (#82) and two wafers (#83 & #84) cycled simultaneously, one with the polished side up and the other with the polished side down.

The quantitative dislocation density fluctuates from run to run but, for a given run, the wafer with the polished side down *always* has more dislocations than the one with the polished side up. The difference in densities from run to run has not yet been explained, but the strong correlation between top and bottom dislocation densities for a given run is preliminary evidence that gravity influences the creation of lattice defects.

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Solutions of Laplace's Equation in a Rectangle with a Large Hole

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EMPLOYING a new technique, the even eigenfunctions of Laplace's equation in a rectangular region with a large central circular hole are derived. The eigenfunctions, satisfying homogeneous outer boundary conditions are generated by the integration on circular paths of the single series form of Green's function obtained with a method somewhat different from the classical techniques. The solutions so obtained are then linearly combined to meet the nonhomogeneous condition at the boundary of the circular hole. The technique is employed to the solutions of the problems of torsion of prismatic bars and steady-state two-dimensional heat conduction. Numerical results are presented for each case.

Consider a region bounded by a rectangle with the length and width of a and b respectively, and containing a large central circular hole of radius ρ_0. Choose x and y axes parallel to the sides a and b of the rectangle with the origin at the center of the circular hole. Consider now the solutions of the nonhomogeneous equation

$$\nabla^2\phi_\ell=\delta(\rho-\rho_1)\cos 2\ell\varphi \tag{1a}$$

$$\nabla^2\equiv\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\quad \rho_1<\rho_0\ \ell=0,1,2,\ldots\quad \xi=\frac{x}{a}\quad \eta=\frac{y}{a} \tag{1b}$$

Received April 7, 1975; revision received July 14, 1975. The computation portion of this project was supported by IUPUI Computing Services.

Index categories: Structural Static Analysis; Heat Conduction.

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in which ρ and φ are the polar coordinates at the origin, and δ is the unit impulse function. Obviously, the solutions for ϕ_ℓ satisfy Laplace's equation for $\rho\geq\rho_0$. In addition, if these solutions satisfy the conditions

$$\phi_\ell=0 \quad\text{at}\quad \xi=\pm\tfrac{1}{2}\quad\text{and}\quad \eta=\pm\frac{b}{2a} \tag{2}$$

then they are the desired eigenfunctions mentioned earlier. To solve Eq. (1), subject to the conditions of Eq. (2), we first obtain Green's function. Let $\bar\phi$ be the solution of

$$\nabla^2\bar\phi=f(\xi,\eta,\xi_1,\eta_1)\quad\text{in rectangle}\quad a\times b \tag{3}$$

in which

$$f(\xi,u,\xi_1,\eta_1)=\dot p^*[\delta(\xi+\xi_1)+\delta(\xi-\xi_1)]$$
$$\text{for}\quad \eta_1>\eta>-\eta_1 \tag{4a}$$

$$\xi_1,\eta_1<\rho_0\quad \dot p=\text{const}\quad\text{(region I)} \tag{4b}$$

$$f(\xi,\eta,\xi_1,\eta_1)=0\quad\text{for}\quad \eta>\eta_1$$
$$\text{and}\quad \eta<-\eta_1\quad\text{(region II)} \tag{4c}$$

Assuming an η independent particular integral of Eq. (3), the appropriate solution $\bar\phi_{\rm I}$ for region I satisfying the homogeneous conditions at $\xi=\pm\frac{1}{2}$ can be written in the form

$$\bar\phi_{\rm I}=\lambda_1(\xi)+\sum_{n=1,3,5\ldots}^{\infty}A_n\cosh n\pi\eta\cos n\pi\xi$$
$$\text{for}\quad \eta_1\geq\eta\geq-\eta_1 \tag{5}$$

where

$$\lambda_1(\xi)=p^*(\xi-\tfrac{1}{2})\qquad\text{for}\qquad \xi>\xi_1 \tag{6a}$$

$$\lambda_i(\xi)=0\qquad\text{for}\qquad -\xi_1<\xi<\xi_1 \tag{6b}$$

$$\lambda_1(\xi)=-p^*(\xi+\tfrac{1}{2})\qquad\text{for}\qquad \xi<-\xi_1 \tag{6c}$$

For region II, we need consider only the solution for $\eta\geq\eta_1$ due to the symmetry. The solution $\bar\phi_{\rm II}$ for this region automatically satisfying the homogeneous conditions at $\xi=\pm\frac{1}{2}$ and $\eta=b/2a$ can now be written as follows

$$\bar\phi_{\rm II}=\sum_{n=1,3,5,\ldots}^{\infty}A_n^*\sinh n\pi\Big(\eta-\frac{b}{2a}\Big)\cos n\pi\xi\quad\text{for}\quad \eta\geq\eta_1 \tag{7}$$

Here in Eqs. (5) and (7), A_n and A_n^* are the unknown constants of integration to be determined from the continuity conditions of the functions $\bar\phi$ and $\partial\bar\phi/\partial\eta$ at $\eta=\eta_1$. Expanding $\lambda_1(\xi)$, defined in Eqs. (6), in Fourier series and employing these continuity conditions, one obtains

$$A_n=-\frac{\beta_n\cosh n\pi(\eta_1-b/2a)}{\cosh(n\pi b/2a)} \tag{8a}$$

$$A_n^*=-\frac{\beta_n\sinh n\pi\eta_1}{\cosh(n\pi b/2a)} \tag{8b}$$

$$\beta_n=4p^*[-(1/n^2\pi^2)\cos n\pi\xi_1-(1/n\pi)\xi_1\sin n\pi\xi_1] \tag{8c}$$

Let us now consider a function $\bar\phi^0$ which satisfies

$$\nabla^2\bar\phi^0=f(\xi,\eta,\xi_1,\eta_1-\epsilon)\quad\text{in rectangle}\quad a\times b \tag{9}$$

in which $\epsilon < \eta_I$ is a small positive number. The solution $\bar{\phi}^0$ of Eq. (9) subject to homogeneous outer boundary conditions can be obtained by changing η_I to $\eta_I - \epsilon$ in $\bar{\phi}$. Adding $\bar{\phi}$ to $\bar{\phi}^0$ and taking the limit as $\epsilon \to 0$ while $p^*\epsilon = 1$, we find the solutions $(\phi_I)_c$ and $(\phi_{II})_c$ for Green's function

$$(\Phi_I)_c = \sum_{n=1,3,5,\ldots}^{\infty} (A_n)_c \cosh n\pi\eta \cos n\pi\xi \quad \text{for } \eta_I \geq \eta \geq -\eta_I \tag{10a}$$

$$(\phi_{II})_c = \sum_{n=1,3,5,\ldots}^{\infty} (A_n^*)_c \sinh n\pi(\eta - b/2a) \cos n\pi\xi \quad \text{for } \eta \geq \eta_I \tag{10b}$$

$$(A_n)_c = -\frac{n\pi\bar{\beta}_n \sinh n\pi(\eta_I - b/2a)}{\cosh (n\pi b/2a)} \tag{10c}$$

$$(A_n^*)_c = -\frac{n\pi\bar{\beta}_n \cosh n\pi\eta_I}{\cosh (n\pi b/2a)} \tag{10d}$$

$$\bar{\beta}_n = \beta_n / p^* \tag{10e}$$

Next, employing the aforementioned Green's function as a kernel, the eigenfunctions ϕ_ℓ can be written in the form

$$\phi_\ell = \int_0^{\varphi^*} (\phi_{II})_c \rho_I \cos 2\ell\varphi_I \, d\varphi_I + \int_{\varphi^*}^{\pi/2} (\phi_I)_c \rho_I \cos 2\ell\varphi_I \, d\varphi_I \tag{11a}$$

$$\rho_I = (\xi_I^2 + \eta_I^2)^{1/2} < \rho_0 \quad \varphi_I = \arctan \eta_I/\xi_I \quad \ell = 0,1,2,3,\ldots \tag{11b}$$

$$\varphi^* = \arcsin \eta/\rho_I \quad \text{for} \quad \eta_I > \eta > -\eta_I \tag{11c}$$

$$\varphi^* = (\pi/2) \quad \text{for} \quad \eta \geq \eta_I \tag{11d}$$

Note that the integrals in Eqs. (11) involve terms such as

$$\cosh n\pi[(\rho_I/a) \sin \varphi_I] \cos n\pi[(\rho_I/a) \cos \varphi_I]$$

To make the integrations analytically possible, we expand these functions in Fourier series. For example, we have

$$\cosh n\pi[(\rho_I/a) \sin \varphi_I] \cos n\pi[(\rho_I/a) \cos \varphi_I] =$$
$$\cos [in\pi(\rho_I/a) \sin \varphi_I] \cos [n\pi(\rho_I/a) \cos \varphi_I] =$$
$$\tfrac{1}{2}\{ \cos[(n\pi\rho_I/a)(\cos \varphi_I + i \sin \varphi_I)] + \cos[(n\pi\rho_I/a)(\cos \varphi_I - i \sin \varphi_I)]\} =$$
$$\tfrac{1}{2}[\cos(\lambda e^{i\varphi_i}) + \cos(\lambda e^{-i\varphi_i})]$$
$$\lambda = \frac{n\pi\rho_i}{a} \quad i = \sqrt{-1} \tag{12}$$

Expanding the terms in the right-hand side of Eq. (12) in powers of $\lambda e^{i\varphi_2}$ and $\lambda e^{-i\varphi_I}$, we obtain

$$\cosh n\pi\left(\frac{\rho_I}{a} \sin \varphi_I\right) \cos n\pi\left(\frac{\rho_I}{a} \cos \varphi_I\right)$$
$$= \tfrac{1}{2}\left[2 - \frac{\lambda^2}{2!}(e^{2i\varphi_I} + e^{-2i\varphi_I}) + \frac{\lambda^4}{4!}(e^{4i\varphi_I} + e^{-4i\nu_I}) - \ldots\right]$$
$$= \sum_{j=1}^{\infty} (-1)^{j+1} \frac{\lambda^{2j-2}}{(2j-2)!} \cos(2j-2)\varphi_I \tag{13}$$

Table 1 Values of nondimensional shear stress $\bar{\tau}_{zx} = \tau z x / G\alpha a$ (G = modulus of shear, α = angle of twist/unit length) at $\rho = \rho_0$ $\varphi = \pi/2$ vs ρ_0/a for the case of $b/a = 1$

ρ_0/a	$\bar{\tau}_{zx}$
0.25	−0.29557
0.30	−0.37909
0.35	−0.47827
0.40	−0.60622

Table 2 Temperature field T vs various ρ/a and φ for the case of $T_{out} = 0$, $T_{in} = 1$, and for $\rho_0/a = 0.25$, $b/a = 1$

φ	$\rho/a = 0.25$	$\rho/a = 0.30$	$\rho/a = .35$	$\rho/a = 0.40$
0	1	0.75323	0.54065	0.34973
$9\pi/68$	1	0.76373	0.56442	0.39256
$\pi/4$	1	0.77213	0.58351	0.42658

It can be shown that, for the points $\eta \geq \eta_I$ corresponding to $\varphi^* = \pi/2$, the results of the integrals in Eq. (11) are greatly simplified, and ϕ_ℓ's are reduced to single series forms.

Applications in Torsion of Prismatic Bars and Steady-State Two-Dimensional Heat Conduction

For the torsion of a prismatic bar with a rectangular cross section having a central circular cavity, we select solutions of Laplace's equation in the form

$$\bar{\psi}(\xi,\eta) = \bar{\psi}^*(\xi,\eta) + \sum_{\ell=0}^{\infty} B_\ell \phi_\ell(\xi,\eta) \tag{14}$$

in which B_ℓ's are certain constants to be determined, and $\bar{\psi}^*$ is the solution for the torsion of a rectangular cross section bar without a cavity.[1] From the properties of the functions[1] $\bar{\psi}^*$ and ϕ_ℓ's it is obvious that $\bar{\psi}$ satisfies the necessary outer boundary conditions automatically. We shall determine the unknown constants B_ℓ' s by employing the inner boundary condition

$$\bar{\psi} = \tfrac{1}{2}\rho_0^2 + K \quad \text{at} \quad \rho = \rho_0 \tag{15}$$

and the condition

$$\oint (\partial\bar{\psi}/\partial\nu) ds = 0 \tag{16}$$

Here K is a constant to be determined along with B_ℓ's and ν and s are the coordinates along the normal and tangential to the path of integration, respectively. Thus, considering the symmetry of the cross section with respect to ξ and η axes, we select m terms in the series of the right-hand side of Eq. (14) and fulfill the conditions of Eq. (15) for γ points in the first quadrang ($\gamma > m$). The γ linear algebraic equations resulting from this procedure together with the equation obtained from Eq. (16) are then solved approximately with the method of least square error[2] to give K and B_ℓ's for $\ell = 0,1,2,3,\ldots m$. Numerical results show that the inner boundary condition is extremely well satisfied. For example, retaining only six terms in the series for ϕ_ℓ, the error in satisfaction of this boundary condition is of the order of magnitude of 10^{-6} for the case of a bar with a square cross section. The problem of steady-state two-dimensional heat conduction in a rectangular region with a hole subject to the temperature $T = 0$ at its outer boundary is similarly solved. For this case, only the functions Σ_ℓ^ϕ are employed and the integral $\oint \partial T/\partial\nu ds$ which is proportional to the heat conducted per unit time, is not zero and does not affect the temperature distribution.

Numerical results for the above two cases are presented in Tables 1 and 2. It is interesting to note that the presented technique can be employed to a noncircular large hole in a rectangular region. It can also be applied in the cases where there are more than one hole involved. Finally, eigenfunctions similar to those derived here may be utilized for the solutions of other two-dimensional problems, such as stress analyses in cylindrical shells containing large holes.

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Bending and Vibration of Transversely Isotropic Two-Layer Plates

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THE use of composite materials in various aerospace and industrial applications has prompted a considerable amount of research on the static and dynamic response of multilayer plates. During the past decade, several authors[1,2] have formulated plate theories by a direct extension of Mindlin's theory[3] for homogeneous plates. Sun and Whitney[4] have shown that laminated plate theories which are based on Kirchhoff's hypothesis, or a simple extension of Mindlin's theory, yield grossly inaccurate natural-frequency predictions for two- and three-layer plates whose layers have widely differing shear rigidities.

In a recent paper,[5] the principle of virtual work was used to derive the equations of motion in an invariant form for an arbitrary three-layered plate. No restrictions were placed on the relative thicknesses, densities, elastic moduli, or symmetries of the layers. The formulation accounts for the shear deformation of each layer as well as the translational and rotational inertia of the composite. Continuity of displacements and stresses was imposed in accordance with a perfect interface bond assumption.

In the current analysis, the previously derived equations will be used to analyze a transversely isotropic two-layer plate by deleting the terms associated with the third layer and neglecting the transverse contraction of the composite. The theory then becomes the two-dimensional analog of Theory II as presented by Sun and Whitney.[4] If we assume that each layer is transversely isotropic, the equations of motion are written in vector notation and can be uncoupled to yield a sixth-order equation in the transverse displacement. By neglecting certain in-plane and rotatory inertia terms, we can obtain a somewhat simpler fourth-order equation, which is very similar to Mindlin's dynamic plate equation with modified stiffness, mass, and inertia coefficients. This equation reduces to Mindlin's formulation if either of the layers is assumed to vanish or the properties of both layers are identical. From virtual work, the natural boundary conditions can also be derived in an invariant form for a plate of arbitrary shape.[5]

To substantiate the differences between the current work and the formulations that are in present use, natural-frequency calculations have been performed for a variety of layer property configurations. The numerical results obtained for transversely isotropic layers indicate that laminated theory[7] based on Kirchhoff's hypothesis will, in general, be in substantial error, even for relatively thin plates, if the ratios of the in-plane-to-transverse shear moduli of the two layers are large.

Governing Equations

The equations of motion are developed for an elastic, transversely isotropic two-layer plate in the state of generalized plane stress. It is assumed that the in-plane displacements vary linearly through the thickness of each layer, however, the cross-sectional rotations of the layers are not necessarily equal. The transverse displacement is assumed to be constant through the plate thickness and the transverse normal stress is assumed to be zero.

The equations of motion can easily be deduced from previously derived equations.[5] When both layers are transversely isotropic then, as before, vector notation can be employed to describe all the equations. It is also possible to uncouple the field equations, which leads to a sixth-order partial differential equation in the transverse displacement w. It can be shown that neglecting certain in-plane and rotatory inertia terms results in the following simplified uncoupled equation

$$B_1 \nabla^4 w + B_2 \nabla^2 w_{,tt} + B_3 w_{,tttt} - \bar{m} \sum_{n=1}^{2} \overset{(n)}{b} w_{,tt} + \sum_{n=1}^{2} \overset{(n)}{b} p_3 + B_4 \nabla^2 p_3 - (B_3/\bar{m}) p_{3,tt} = 0 \qquad (1)$$

where

$$B_1 = -[\sum_{n=1}^{2} (\overset{(n)}{b} \overset{(n)}{h})^2 + \overset{(1)}{b} \overset{(2)}{b} (4 \sum_{n=1}^{2} \overset{(n)}{h}{}^2 + 6 \overset{(1)}{h} \overset{(2)}{h})]/12$$

$$B_2 = \{ \sum_{n=1}^{2} \overset{(n)}{\rho} \overset{(n)}{b} \overset{(n)}{h}{}^3 + \overset{(2)}{\rho} \overset{(1)}{b} \overset{(2)}{h} (6 \overset{(1)}{h}{}^2 + 9 \overset{(1)}{h} \overset{(2)}{h} + 4 \overset{(2)}{h}{}^2)]$$

$$- \overset{(1)}{\rho} \overset{(2)}{b} \overset{(1)}{h}{}^2 (2 \overset{(1)}{h} + 3 \overset{(2)}{h})$$

$$+ (\bar{m}/h_T) [\sum_{n=1}^{2} (\overset{(n)}{b}{}^2 \overset{(n)}{h}{}^2 / \overset{(n)}{a}) + \overset{(1)}{b} \overset{(2)}{b}] \{ (\overset{(1)}{a} \overset{(2)}{h}{}^2$$

$$+ \overset{(2)}{a} \overset{(1)}{h}{}^2)]/\overset{(1)}{a} \overset{(2)}{a} + 3 \overset{(1)}{h} \overset{(2)}{h} (\overset{(1)}{a} + \overset{(2)}{a})$$

$$/ \overset{(1)}{a} \overset{(2)}{a} \}]\}/12$$

$$B_3 = (\bar{m}/12h_T) \{ - \sum_{n=1}^{2} \overset{(n)}{\rho} \overset{(n)}{b} \overset{(n)}{h}{}^3 / \overset{(n)}{a} - (\overset{(2)}{\rho} \overset{(1)}{b} \overset{(2)}{h} / \overset{(2)}{a})$$

$$(6 \overset{(1)}{h}{}^2 + 9 \overset{(1)}{h} \overset{(2)}{h} + 4 \overset{(2)}{h}{}^2)] + (\overset{(1)}{\rho} \overset{(2)}{b} \overset{(1)}{h}{}^2$$

$$/ \overset{(1)}{a}) (2 \overset{(1)}{h} + 3 \overset{(2)}{h}) \}$$

$$B_4 = - \{ \sum_{n=1}^{2} \overset{(n)}{b}{}^2 \overset{(n)}{h}{}^2 / \overset{(n)}{a}$$

$$+ \overset{(1)}{b} \overset{(2)}{b} [4 \sum_{n=1}^{2} \overset{(n)}{h}{}^2 / \overset{(n)}{a}$$

$$+ 3 \overset{(1)}{h} \overset{(2)}{h} \sum_{n=1}^{2} \overset{(n)}{a} / \overset{(1)}{a} \overset{(2)}{a}] \}/12h_T$$

Received April 7, 1975; revision received July 9, 1975.

Index categories: Structural Composite Materials (including Coatings); Structural Dynamic Analysis.

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